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This programed booklet is designed for the engineering studet who understands and can use vector and unit vector notation, components of avector, parallel law of vector addition, and the dot product of two vectors. Conten begins with work done by a force in moving a body a certain distance along some pth. For each of the examples and problem solving discussions there is given a pat whose equation in terms of the coordinates is known and a mathematical expressionfor the direction and magnitude of the applied force. Through such a series of exames and problem solving discussions, the student is led to the idea of the line integal of a vector. (RP)

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LINE INTEGRAL O' A VECTOR

by

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BEFORE YOU BEGIN

This programmed material assumes that you understand and can use the following:

1. Vector notation, $\vec{V}$;
2. Unit vector notation, $\vec{u_x}$, $\vec{u_y}$ or $\vec{u_r}$;
3. Components of a vector;
4. Parallegram law of vector addition;
5. The dot product of two vectors, $\vec{A} \cdot \vec{B}$.

If you find that you have any difficulty with these concepts, either now or as you are going through the program, get some help. Your instructors will be glad to review these topics with you.

1. Line Integrals

You are all familiar with the notion of the work done by a force in moving a material body a certain distance along some path. You should also know that the direction of the force and the direction of the path of motion are not necessarily the same. Thus, the work done is not simply the product of force times distance moved, but the projection of the force along the path of motion times the distance moved. Thus, suppose a constant force F, as shown in Fig. 1, moves a body of mass m a distance d along the guide rail of some device. The work done by the force will be

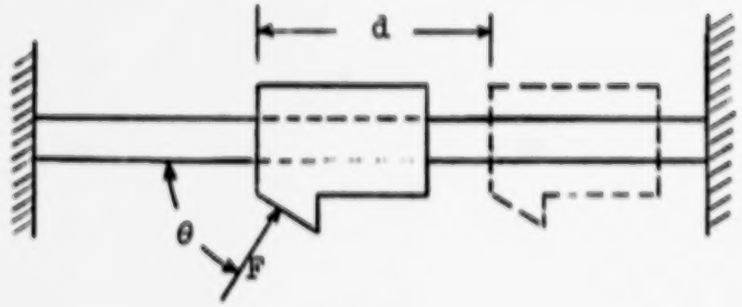


Fig. 1

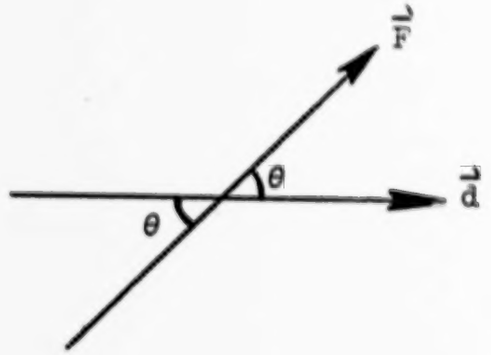
Work = _____

Answer:

Work = (F cos θ) times d = Fdcosθ.

Now force is a <u>vector</u> quantity and so is <u>displacement</u>. In the answer just given F and d are the <u>magnitude</u> of these respective vectors, which are written in vector notation as $\vec{F}$ and $\vec{d}$.

Can you write the expression for work as a mathematical operation involving these vectors? Do it and name the operation.



Work = _____

Answer:

Work = $\vec{F} \cdot \vec{d}$

Work is the dot product of $\vec{F}$ and $\vec{d}$.

If the angle between $\vec{F}$ and $\vec{d}$ is θ, the dot product $\vec{F} \cdot \vec{d}$ is defined as

$\vec{F} \cdot \vec{d} =$ _____

Answer:

$\vec{F} \cdot \vec{d} = Fd\cos\theta$

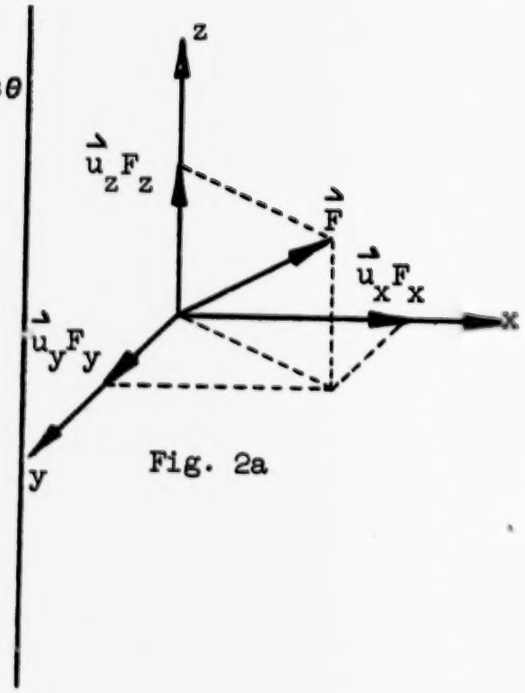


Fig. 2a

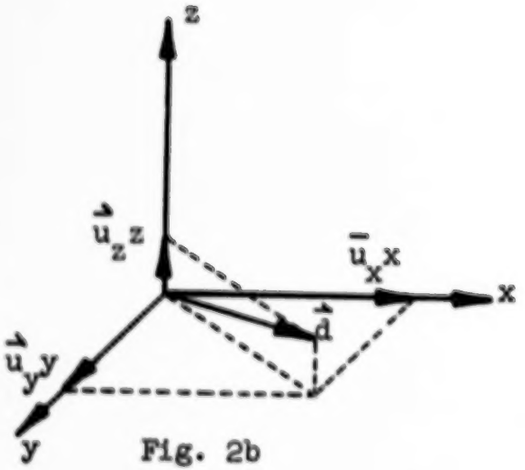


Fig. 2b

If $\vec{F}$ and $\vec{d}$ in Fig. 2 are written in rectangular coordinates as

$$\vec{F} = \vec{u}_x F_x + \vec{u}_y F_y + \vec{u}_z F_z$$

$$\vec{d} = \vec{u}_x x + \vec{u}_y y + \vec{u}_z z$$

then the dot product in terms of the components of the two vectors becomes

$$\vec{F} \cdot \vec{d} = _____$$

Answer:

$$\vec{F} \cdot \vec{d} = xF_x + yF_y + zF_z$$

Thus, suppose:

$$\vec{F} = \hat{u}_x 10 + \hat{u}_z 4$$

You can write a general expression for $\vec{d}$

$$\vec{d} = _____$$

Then the expression for the dot product $\vec{F} \cdot \vec{d}$ is

$$\vec{F} \cdot \vec{d} = _____$$

Answer:

$$\vec{d} = \vec{u}_x x + \vec{u}_y y + \vec{u}_z z$$

$$\vec{F} \cdot \vec{d} = 10x + 4z$$

(The dot product is the arithmetic sum of the products of the components along each coordinate axis.)

In the illustration shown in Fig. 1 there were some characteristics of the system that made the problem relatively simple. As the body moved along the guide rail, the magnitude of the force remained constant and so did its _____. Furthermore, the direction of the path also _____ _____.

Answer:

direction

remained constant.

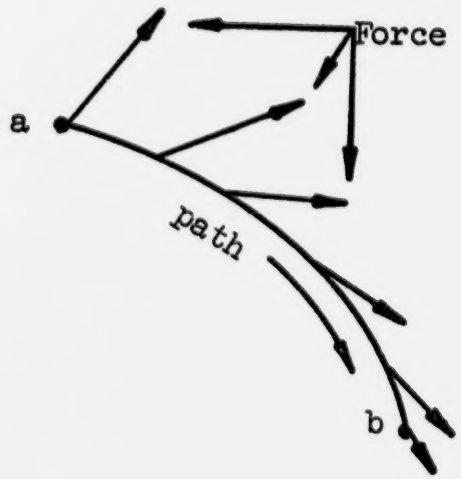


Fig. 3

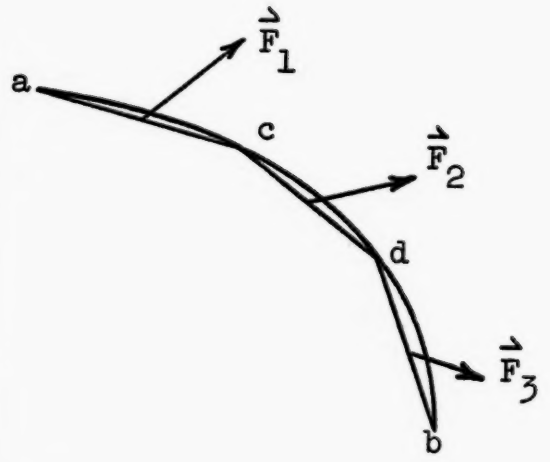
In a more general problem neither the _____ nor the _____ of the force need remain constant. Furthermore, the path may not be a straight line as it was in Fig. 1.

Consider the situation in Fig. 3 which shows a curved path and a force which varies in both magnitude and direction along the path, as illustrated by the changes in length and orientation of the arrows. In order to find the work done by the force in moving an object along this path, we make a number of observations and assumptions:

1. We observe that a curve or segment of a curve can be approximated by a straight line. The approximation gets better and better as the length of the curve segment is made smaller and smaller.
2. We assume that the changes of the direction and magnitude of the force along the path are not violent but smooth.

On the basis of these thoughts, describe in general terms the steps in a procedure by which the work done by the force $\vec{F}$ can be approximated along the path from a to b.

Answer: 1. The curved path can be approximated by a sequence of straight line segments. (The figure shows only 3 for clarity.) 2. An "average" force, constant in direction and magnitude, is assumed to act along each line segment. 3. The work done along each line segment can therefore be found by the previously discussed procedure ($\vec{F} \cdot \vec{d}$). 4. The total work is approximated by adding the work done along each segment. (Your answer will be differently worded but should contain all four of the points listed. If it does not, construct another statement, without looking at this answer again.)



In terms of the preceding diagram, let $\overrightarrow{\Delta s_1}$, $\overrightarrow{\Delta s_2}$, and $\overrightarrow{\Delta s_3}$ be the vectors representing the displacement from a to c, c to d and d to b, respectively. Then,

work done by the force from a to c = $\overrightarrow{F_1} \cdot \overrightarrow{\Delta s_1}$

work done by the force from c to d = _____

work done by the force from d to b = _____

total work done from a to b along the straight lines = _____

Answer:

$\overrightarrow{F_2} \cdot \overrightarrow{\Delta s_2}$

$\overrightarrow{F_3} \cdot \overrightarrow{\Delta s_3}$

$$\text{total work} = \overrightarrow{F_1} \cdot \overrightarrow{\Delta s_1} + \overrightarrow{F_2} \cdot \overrightarrow{\Delta s_2} + \overrightarrow{F_3} \cdot \overrightarrow{\Delta s_3}$$

$$= \sum_{k=1}^{3} \overrightarrow{F_k} \cdot \overrightarrow{\Delta s_k}$$

(You may not have written the answer in the last form, which is OK. But, study this form so you can use it on problems.)

Thus, the total work is the sum of three terms, each one of which is the dot product of a force and an appropriate displacement vactor. If θ_k is the angle between the kth force and the kth displacement, use the definition of a dot product to rewrite the last expression in the summation form.

$$\text{total work} = \sum_{k=1}^{3} \vec{F_k} \cdot \vec{\Delta s_k} = _____$$

Suppose that the path from a to b is divided into n segments instead of just 3. Write and expression similar to the last one for the total work done by the force in moving an object from a to b, along these line segments as a dot product of vectors, and also in expanded form.

work done = _____ (Dot Product)

= _____ (Scalar form)

Answer:

$$\text{work done} = \sum_{k=1}^{n} \vec{F}_k \cdot \vec{\Delta s}_k$$

$$= \sum_{k=1}^{n} F_k \cos\theta_k \, \Delta s_k$$

(You probably had the order of $\cos\theta_k$ and Δs_k interchanged. The only reason for writing it in this order is for future convenience.)

This expression is the work done by the force in moving an object from a to b along _____. It is _____ (exactly/approximately) equal to the work done in moving the object from a to b along the curve.

Answer:

the line segments (n of them).

approximately

A question that must be settled is how to find the "average" force to use for each line segment. This is where the assumption of the smoothness of variation of the force comes in.

If the force changes gradually (both in magnitude and direction) from point to point along the path, then the error made in selecting the force at the center of each segment, say, as the "average" force to use will not be large, especially if the length of each segment is small.

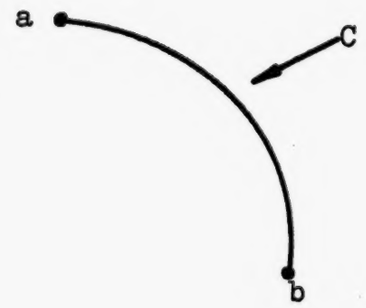
Next, suppose that the number of segments into which the curve is divided is allowed to increase. In the limit, as the number of segments goes to infinity and the length of each one goes to zero, what becomes of the summation? What becomes of the error which results from calculating the work along the straight line segments instead of the curve?

Answer:

As the number of segments n goes to infinity, the summation becomes an integral.

The error approaches zero and the integral gives the work done exactly.

The resulting integral is called a <u>line integral</u>. Specifically, it is the line integral of the force $\vec{F}$.



The work done by the force $\vec{F}$ in moving an object along the curve labeled C from point a to point b is written as

$$W = \int_a^b \vec{F} \cdot \vec{ds}$$

The right-hand side of this expression is called _____.

Answer:

the line integral of $\vec{F}$.

(You may have said just line integral; this is not incorrect, but it is incomplete.)

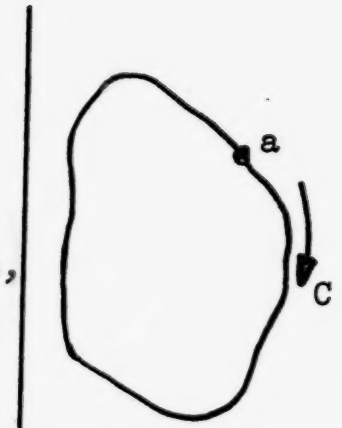


Fig. 4

Very often, the path along which it is desired to integrate is a closed one, as shown in Fig. 4. That is, we start at some point a and we integrate all the way around a closed curve back to a. The result is then called a closed line integral.

In the previous case, when integrating from a to b along an open curve, there was only one way to reach b, starting at a. For a closed line integral there are two ways in going around the curve: clockwise and _____.

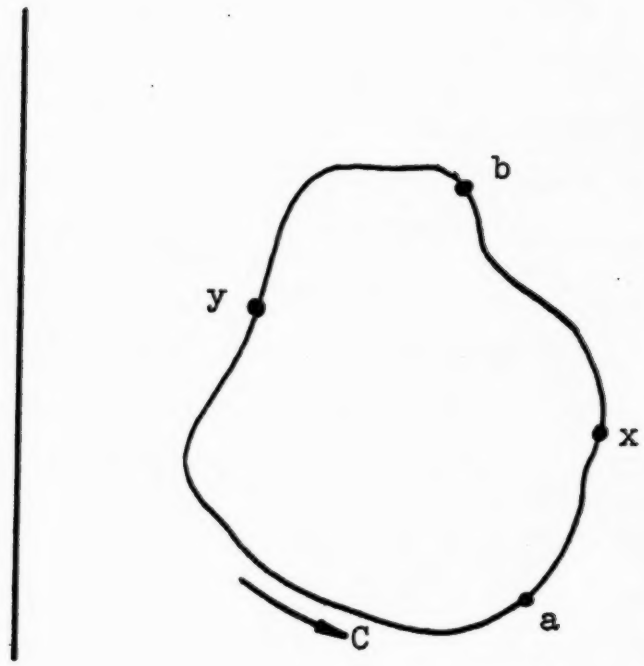
In order to indicate that a line integral is closed and to indicate the direction of integration we use the following special notation on the integral sign: $\oint_C$. This symbol indicates a closed line integral taken counterclockwise; the letter C under the integral sign indicates that the integration is performed along a closed curve labeled C.

Write an expression for the closed line integral of $\vec{F}$ clockwise around a curve C.

W = _____

Answer:

$$W = \oint_C \vec{F} \cdot \vec{ds}$$



A closed path composed of two segments is shown in the figure: a segment from a to b via x and another from b back to a via y.

The line integral of $\vec{F}$ from a to b can be added to that from b back to a to yield a closed line integral around the entire curve C.

Write this underlined sentence in mathematical form.

_____

Answer:

$$\underset{\text{through x}}{\int_a^b} \vec{F} \cdot \vec{ds} + \underset{\text{through y}}{\int_b^a} \vec{F} \cdot \vec{ds} = \oint_c \vec{F} \cdot \vec{ds}$$

Consider the line integral of a force $\vec{F}$ from a to b along a curve C. Suppose the direction of integration is reversed and we take instead the line integral of $\vec{F}$ from b to a. How are these two integrals related? Write it in mathematical form.

Answer:

One is the negative of the other.

$$\int_a^b \vec{F} \cdot \vec{ds} = - \int_b^a \vec{F} \cdot \vec{ds}$$

along same curve.

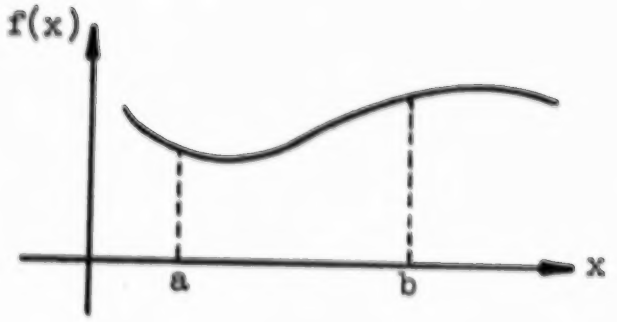


Fig. 5

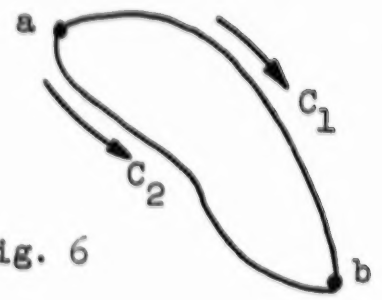


Fig. 6

In this result, it is necesary to stipulate that the integration is along the same curve. This may seemsurprising because you may recall that the definite integral of a function of a rea variable depends only on the limits of integration. Thus, Fig. 5 shows a sketch of a function f(x) against x. The integral of f(x) from a to b is the area under the curve. For a given function f(x) this area will depend only on the limits a and b.

However, when dealing with the line integral of a vector between two points a and b, there are an infinite number of paths that can be chosen, two of which are shown labeled C_1 and C_2 in Fig. 6. The question arises as to whether or not the line integral of a vector btween the two end points is the same if the path taken is along C_1 or along C_2. It is not possible to answer this question in a general way with the state of knowledge assumed here. We can, however, evaluate the line integral in some specific cases and observe the results.

As a first example consider the situation shown in Fig. 7. It is assumed that there is a force throughout a region of space whose direction at any point is radially outward from some center. Let $\vec{u}_r$ be a unit vector in the radial direction, and r the distance from the center. The force is then assumed to be given by

$$\vec{F} = \vec{u}_r \frac{K}{r^2}$$

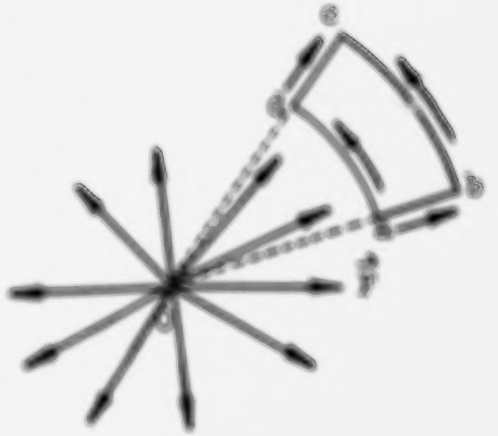


Fig. 7

It is desired to find the line integral from a to c along two paths: one from a to b to c, the other from a to d to c, in Fig. 7. The segments from a to b and from d to c are on radial lines from the center. The curves from b to c and from a to d are arcs of circles with centers at point O. Let's consider each of these four segments separately.

Using the definition, find the value of the line integral of $\vec{F}$ from b to c and from a to d, in the directions indicated.

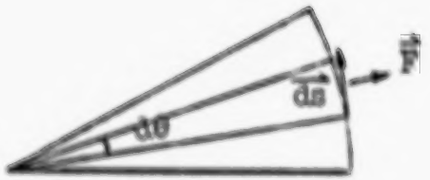
$$\int_b^c \vec{F} \cdot \vec{ds} = _____$$

$$\int_a^d \vec{F} \cdot \vec{ds} = _____$$

Answer:

$$\int_b^c \vec{F} \cdot \vec{ds} = 0$$

$$\int_a^d \vec{F} \cdot \vec{ds} = 0$$



Both line integrals are zero because the direction of the force is at right angles to the direction of the path element $\vec{ds}$. Hence, the dot product $\vec{F} \cdot \vec{ds} = Fds\cos\theta = 0$.

Now turn to the segment from a to b, in Fig. 7. Write an expression for $\vec{ds}$ along this line segment in terms of the unit vector in the radial direction.

$$\vec{ds} = _____$$

How is the direction of $\vec{ds}$ related to that of $\vec{F}$ along this segment of path from a to b? _____

Evaluate the line integral of $\vec{F}$ from a to b.

$$\int_a^b \vec{F} \cdot \vec{d} = _____$$

Answer:

$\vec{ds} = \vec{u}_r\, dr$ (You may have written the magnitude as ds. Since the path is in the direction of the radius r, then ds = dr.)

$\vec{F}$ and $\vec{ds}$ are in the same direction. Hence,

$$\int_a^b \vec{F} \cdot \vec{ds} = \int_a^b F\, dr = \int_a^b \frac{K}{r^2}\, dr = \boxed{-\frac{K}{r}\Bigg|_{\text{point a}}^{\text{point b}}}$$

See next page for further discussion of this expression.

You should <u>not</u> simply put r = b and r = a as the limits because these symbols just designate the <u>locations</u> of the points on the path. They are <u>not</u> intended to specify the coordinates of these points. If we let r_a and r_b represent the radial distances from the origin to a and b, then the expression for the line integral from a to b will become

---

Answer:

$$\int_a^b \vec{F} \cdot \vec{ds} = K \left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

Finally, the segment of path from d to c. From similar considerations (specifically, the direction of $\vec{ds}$ and $\vec{F}$), evaluate the line integral of $\vec{F}$ along this path. (Use r_d and r_c to represent the coordinates of the points d and c.)

Answer:

$$\int_d^c \vec{F} \cdot \vec{ds} = \int_d^c F \, dr \qquad \text{(because } \vec{ds} = \vec{u}_r \, ds \text{ is in the same direction as } \vec{F}.\text{)}$$

$$= \int_d^c \frac{K}{r^2} \, dr = K\left(\frac{1}{r_d} - \frac{1}{r_c}\right)$$

You can now complete the evaluation of the line integral of $\vec{F}$ from a to c along the two paths a⟶b⟶c and a⟶d⟶c. Looking at Fig. 7 to observe the relative values of r_a, r_d and r_b, r_c, state whether these two line integrals are the same or different. Justify your conclusion.

Answer:

The same, because

$r_a = r_d$ and $r_b = r_c$.

We have now determined that the line integral of $\vec{F}$ from a to c is the same along each of two paths in Fig. 7.

What can you say about the value of the closed line integral around the path abcda in Fig. 7?

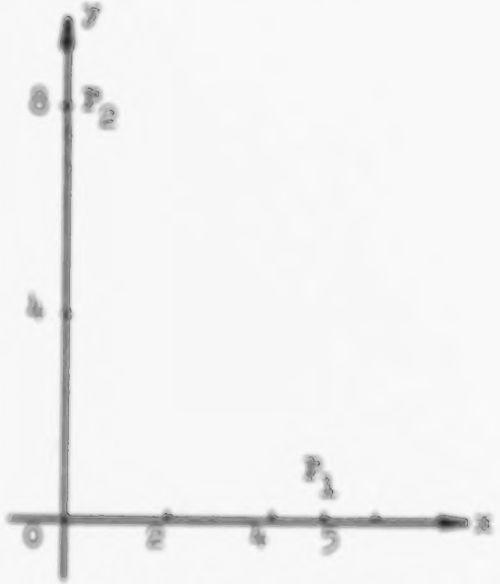
Write it as a mathematical statement.

Answer:

The closed line integral of $\vec{F}$ around the path abcda is zero.

$$\oint \vec{F} \cdot \vec{ds} = 0.$$

Caution: This still does not answer the general question as to whether the line integral of $\vec{F}$ from a to c along any path will be the same. In order to answer that question somewhat more sophisticated mathematics would be required than we are using.



distances in meters

Now here is a problem for you to do. A radial force of the type just discussed is given by

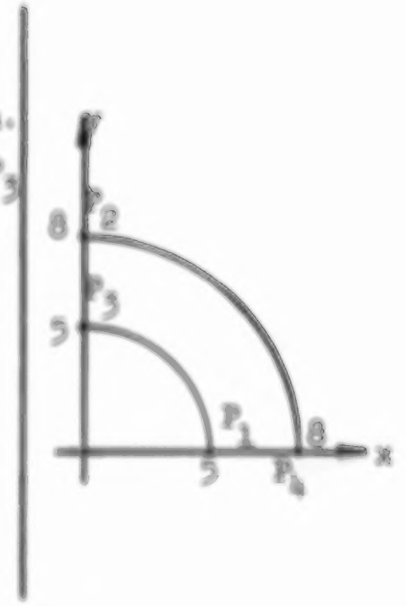
$$\vec{F} = \hat{u}_r \frac{10}{r^2} \text{ newtons}$$

where r is the distance from the origin.

Find the work done by this force in moving an object from the point P_1 to the point P_2 in the diagram.

Answer: The work is calculated as follows:

Drav a circular arc of radius 5 units around the origin. A path from P_1 to P_3 is formed by the circular path from P_1 to P_3 plus the straight path up the y axis from P_3 to P_2. Along the circular arc $\vec{ds}$ and $\vec{F}$ are at right angles and so contribute nothing to the line integral. Along the vertical path $\vec{ds}$ and $\vec{F}$ are in the same direction.



$$\vec{F} = \vec{u}_y \frac{10}{y^2}, \quad \vec{ds} = \vec{u}_y \, dy$$

$$\text{Work} = \int_{P_3}^{P_2} \vec{F} \cdot \vec{ds} = \int_5^8 \frac{10}{y^2} \, dy = -\frac{10}{y}\Big|_5^8 = \frac{5}{4} \text{ joules.}$$

An alternate approach would be to go out along the x axis from P_1 to P_4 and from there along a circular arc to P_2. Do it (if you didn't) and verify that the same answer results.

As another illustration of the evaluation of a line integral, consider the situation depicted in Fig. 8.

A force $\vec{F}$ is directed in the y direction and its magnitude varies with x, as shown. It is desired to evaluate the line integral of $\vec{F}$ from a to c along two paths: path 1 goes from a to b up the y axis, then from b to c parallel to the x axis; path 2 goes directly from a to c in a straight line.

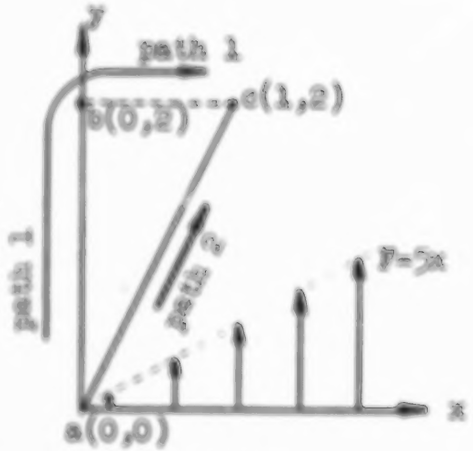


Fig. 8

For each of these paths we must find the values of the magnitude of $\vec{F}$, the magnitude of $\vec{ds}$ and the cosine of the angle between $\vec{F}$ and $\vec{ds}$.

A vector $\vec{ds}$ is in the x-y plane. Along an arbitrary path in the plane, $\vec{ds}$ can be expressed in rectangular components, by considering the small triangle shown and using the unit vectors $\vec{u}_x$ and $\vec{u}_y$ as:

$\vec{ds}$ = _____

Answer :

$$\vec{ds} = \vec{u_x}\, dx + \vec{u_y}\, dy .$$

Note: Before completing the solution of the problem on the last page, let's examine $\vec{ds}$ further. This page, and the two pages following, will provide you some additional practice in writing expressions involving $\vec{ds}$, $|\vec{ds}|$, dx and dy.

The magnitude of $\vec{ds}$ along any arbitrary path can be found from the preceding expression on page 30 to be:

$ds = |\vec{ds}| =$ _____

The angle which such a $\vec{ds}$ makes with the x-axis has a cosine which equals:

cosine of angle between $\vec{ds}$ and x-axis = _____

Similarly, the angle which $\vec{ds}$ makes with the y-axis has a cosine which equals:

_____

Answer:

$ds = \sqrt{dx^2+dy^2}$, which can also be written as $dx\sqrt{1+(\frac{dy}{dx})^2}$ or $dy\sqrt{1+(\frac{dx}{dy})^2}$

cosine of angle $1 = \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2+dy^2}} = \frac{1}{\sqrt{1+(\frac{dy}{dx})^2}}$

sine of angle $2 = \frac{dy}{ds} = \frac{dy}{\sqrt{dx^2+dy^2}} = \frac{1}{\sqrt{1+(\frac{dx}{dy})^2}}$

You may have written any one of these forms, which is OK. If you did not write <u>all</u> of them, look them over to see how they follow from each other. You can probably tell which ones we will be most likely to use, later.

For example, consider the straight line path in the x-y plane shown in Fig. 9. The expression dy/dx appearing under the radical in the last few results is the _____ of this line.

If $\vec{ds}$ is a vector increment along this path, write its magnitude, ds, as a numerical multiplier times dx, and also as a numerical multiplier times dy.

ds = _____ dx

ds = _____ dy

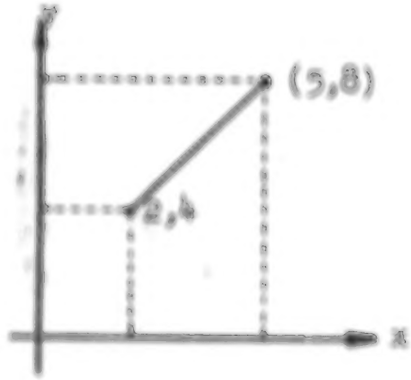


Fig. 9

Answer:

slope

$$ds = \frac{5}{3} dx$$

$$ds = \frac{5}{4} dy$$

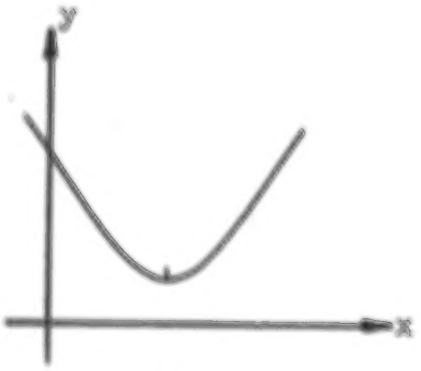


Fig. 10

To take a more complicated example, consider the path consisting of the parabola shown in Fig. 10 and having the equation

$$y = (x-2)^2 + 1$$

This time the slope of the path is not a constant for all values of x. Again express ds as "something" times dx, and also as "something" times dy.

$$ds = _____\ dx$$

$$dx = _____\ dy$$

Answer:

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+4(x-2)^2}\,dx$$

$$ds = \sqrt{1+\frac{1}{4(x-2)^2}}\,dy = \frac{\sqrt{1+4(x-2)^2}}{2(x-2)}\,dy$$

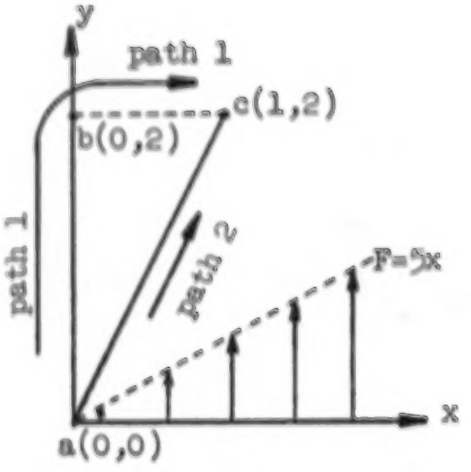


Fig. 8 (repeated)

Now, let's turn back to our problem of Fig. 8 after this slight digression.

Considering first path 2, write expressions for the magnitude of the force $\vec{F}$, the magnitude of $\vec{ds}$ and the cosine of the angle between them for any point along the path.

$F =$ _____

$ds =$ _____ (Write this both in terms of dx and in terms of dy.)

$=$ _____

$\cos \theta =$ _____

2

Answer:

$F = 5x$

$ds = \sqrt{5}\,dx = \frac{\sqrt{5}}{2}\,dy$

$\cos\theta = \frac{2}{\sqrt{5}}$ (Since the force $\vec{F}$ is parallel to the y axis, the angle of $\vec{ds}$ is the angle the path makes with the y axis.)

Using these results, the line integral from a to c along path 2 is calculated to be

$$\int_{a,\ \text{path 2}}^{c} \vec{F} \cdot \vec{ds} =$$

Answer: $\int_{a}^{c} \vec{F} \cdot \vec{ds} = 5$

path 2

This is calculated as follows:

1. $\vec{F} \cdot \vec{ds} = F \cos\theta \, ds = (5x)(\frac{2}{\sqrt{5}})(\sqrt{5}\,dx) = 10x\,dx.$

2. Since the integration is with respect to x, the coordinates of the points a and c are the values of x at these points. Thus,

$$\int_{a}^{c} \vec{F} \cdot \vec{ds} = \int_{0}^{1} 10x\,dx = 5x^2 \Big|_{0}^{1} = 5$$

3. Since x and y along the path are related by the straight line $y = 2x$, $dy = 2dx$, $\vec{F} \cdot \vec{ds}$ can also be written as a function of y Thus, $10x\,dx = 10(\frac{y}{2})(\frac{dy}{2}) = \frac{5}{2}\,y\,dy$. Also, the coordinates of points a and c in terms of y are 0 and 2. So,

$$\int_{a}^{c} \vec{F} \cdot \vec{ds} = \int_{0}^{2} \frac{5}{2}\,y\,dy = \frac{5}{4}\,y^2 \Big|_{0}^{2} = 5$$

which agrees with the previously found value.

Next turn to path 1. This path has two segments. Write the values of F, ds and cos θ for each of these segments.

	F	ds	cos θ
a to b:	___	___	_____
b to c:	___	___	_____

From these values, evaluate the line integral of $\vec{F}$ along path 1.

$$\int_a^c \vec{F} \cdot \vec{ds} = _____$$

path 1

Answer:

	F	ds	$\cos\theta$
a to b:	0	dy	1
b to c:	5x	dx	0

From a to b the path is along the y axis, so ds = dy and the directions of $\vec{F}$ and $\overrightarrow{ds}$ are the same. However, along the y axis x = 0 and so F = 5x is also zero. This part of the path thus contributes nothing to the line integral. Furthermore, from b to c, $\vec{F}$ and $\overrightarrow{ds}$ are at right angles and so the cosine of the angle between them is zero. Hence,

$$\int_a^c \vec{F} \cdot \overrightarrow{ds} = 0.$$

Using the previous results, specify the value of the closed line integral of $\bar{F}$ around the path a-c-b-a. Specify its value around the closed path a-b-c-a also. Write these in mathematical form.

Answer:

$$\oint_{acba} \vec{F} \cdot \vec{ds} = 5 - 0 = 5$$

$$\oint_{abca} \vec{F} \cdot \vec{ds} = 0 - 5 = -5$$

We have thus found an example in which the line integral of a vector between two points is different along two different paths. Hence, the closed line integral does not vanish. Although there isn't enough generality in the last two examples to arrive at a conclusion concerning when the closed line integral will vanish and when it won't, it must depend on the specific manner in which the vector $\vec{F}$ varies with coordinates.

As a final example of the analytical evaluation of line integrals, consider the situation shown in Fig. 11. A body of mass m is to be raised against the force of gravity from point a to point c. Calculate the amount of work required to move the body along each of the two paths: path 1 directly from a to c in a straight line and path 2 from a vertically to b then horizontally to c. Decide beforehand whether the two are the same or different.

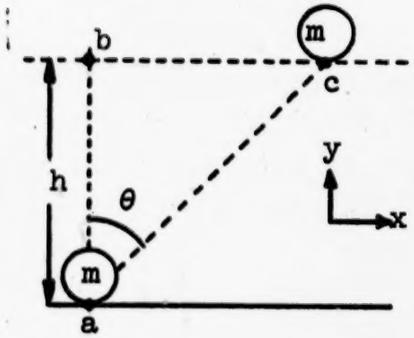


Fig. 11

Answer: Assuming that the distance from b to c is small enough that the curvature of the earth is not a factor, the force of gravity will be a constant equal to the weight of the mass (which is mg, where g is the acceleration of gravity) and will be directed vertically downward. To overcome this force we must exert an equal force vertically upward. Along the path from a to b, $F = mg$, $ds = dy$ and $\cos\theta = 1$; hence,

$$\int_a^b \vec{F} \cdot \vec{ds} = \int_0^h mg\, dy = mgh.$$

From b to c $\vec{ds}$ is in the x direction while the force is in the y direction. Hence, $\cos\theta = 0$ and there is no contribution from this part of the path.

As for the direct path from a to c, the force is still constant at the value mg, but ds is now given by $ds = dy/\cos\theta$. Hence

$$\int_a^c \vec{F} \cdot \vec{ds} = \int_a^c (mg)(\cos\theta)\left(\frac{dy}{\cos\theta}\right) = \int_0^h mg\, dy = mgh.$$

Thus, the work done against the force of gravity is the same in taking the body from a to c along either path.

In each of the examples of line integrals so far considered, there has been a path whose equation in terms of the coordinates we have known. We have also known a mathematical expression for the direction and magnitude of the force. However, suppose an equation for the path is not known but the path is given graphically, as in Fig. 12. Suppose, further, that all that is given about the force is a numerical tabulation of its magnitude and direction at various points along the path (shown by dots in Fig. 12) obtained by measurements.

Describe a procedure for obtaining a value of the line integral in such a case.

Fig. 12

(Magnitude and direction of F known at each point shown.)

Answer:

The line integral was first introduced as the limit of a sum of terms obtained by approximating a curve by a set of line segments. So, to get an approximate value for a line integral, approximate the graphically given path by a number of line segments. Choose the line segments so that each point at which the force is measured falls on a line segment. Assume the force stays constant in magnitude and direction over each segment. Take the product $F_i \Delta s_i \cos \theta_i$ for the ith segment, then add them all to get an approximation of the line integral.

An object is moved by a force along the path in the x-y plane shown in Fig. 13. The magnitude and angle of the force are measured at the four points labeled on the curve and are tabulated here.

Point	$\lvert\vec{F}\rvert$ in newtons	Angle of $\vec{F}$ with respect to x-axis
a	12	70°
b	8	40°
c	6	30°
d	4	50°

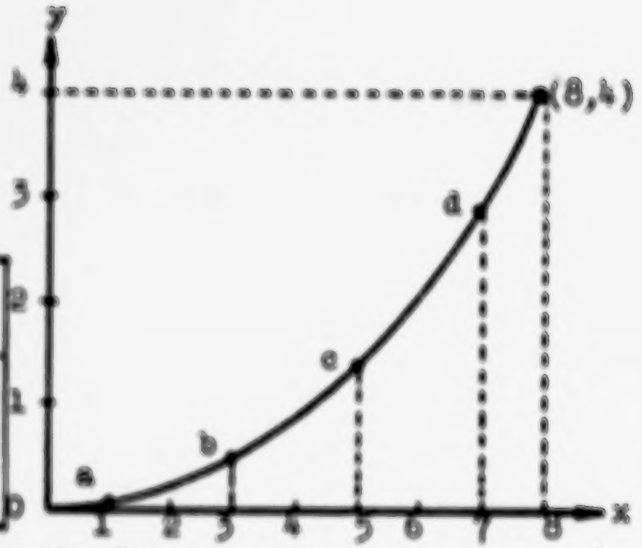


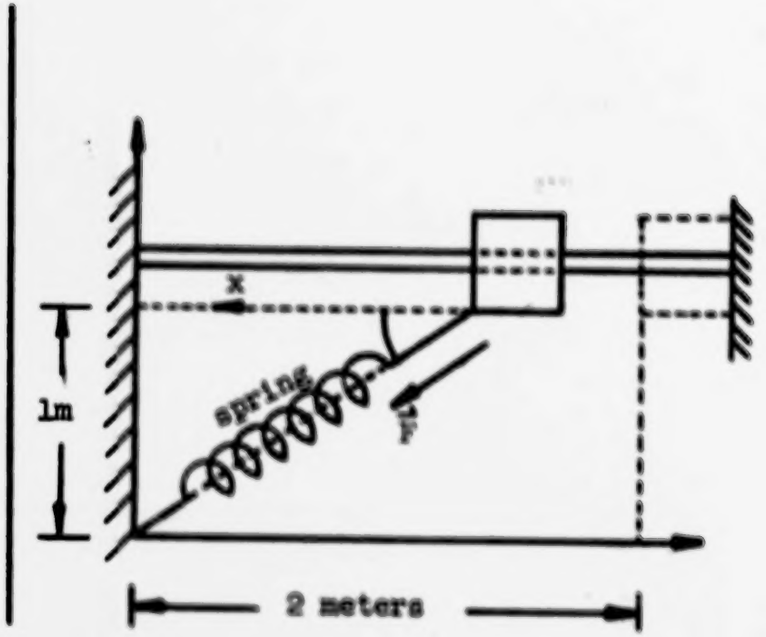
Fig. 13

Find an approximate value of the work done by the force.

1. Do this by approximating the curve with 4 line segments.
2. Do it also by approximating the curve with only 2 segments. In this case decide on some "average" value of force to use for each segment, both in magnitude and angle, and discuss your choice.

Answer:

No verification will be given here. Compare the two values you have obtained.



Here is a final problem for you to do. A body of negligible weight moves along a frictionless straight guide rail. It is pulled along by a stretched spring, as shown in the diagram, whose unstretched length is 1 meter. The force which the spring exerts is proportional to its elongation. That is,

$$F = 100s$$

where s is the total, stretched length of the spring minus its unstretched length. As the body moves, the elongation of the spring changes and so also does the force. The direction of the force also changes.

Using the terminology and notation of this booklet, find the work done by the spring in moving the object from the far end of the rail to the other. To start, specify the numerical values of s and cos θ when the body is at each end of the path.

	Right-hand end	Left-hand end
$s =$	_____	____
$\cos \theta =$	_____	____

Answer:

	Right-hand end	Left-hand end
$s =$	$\sqrt{5} - 1 = 1.236$	0

$$\cos\theta = \frac{2}{\sqrt{5}}$$

(The elongation of the spring, when the body is at any point x, is $s = \sqrt{x^2 + 1} - 1$. At the right-hand end $x = 2$.)

Now complete the problem to find the work done by the spring force, using the techniques you have been practicing.

Answer:

The magnitude of the force when the body is at any point x is $100(\sqrt{x^2+1} - 1)$.

The cosine of θ at this point is $x/\sqrt{x^2+1}$. Since the path is in the x direction, ds = dx. Hence

$$\int_{\text{one end}}^{\text{the other end}} \vec{F} \cdot \vec{ds} = \int_0^2 100(\sqrt{x^2+1} - 1) \frac{x}{\sqrt{x^2+1}}\, dx$$

$$= 100 \int_0^2 (x - \frac{x}{\sqrt{x^2+1}})\, dx = 100(\frac{x^2}{2} - \sqrt{x^2+1}) \Big|_0^2$$

= 78 joules.